Announcements

1) HW due date extended to tomorrow

Example 1:

$$\int \ln(x^5) - \frac{\arctan(2x)}{x^2}$$

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{n} \quad (-1)^{n+1} \quad (-1)^n (x-1)^n$$

$$\int \frac{1}{x} = \int \frac{1}{1-(1-x)} = \int \sum_{n=0}^{\infty} (1-x)^n$$

$$\ln(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{x^m}{m}$$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(x^5) = 5\ln(x)$$

$$= 5\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}(-1)^{n+1}$$

$$\frac{\arctan(2x)}{x^2} = \frac{1}{x^2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{2n+1}$$

$$= \frac{2}{x} + \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}\, 2^{2n+1}}{2n+1}$$

$$\int \ln(x^5) - \frac{\arctan(2x)}{x^2}\, dx$$

$$\int \frac{\ln(x^5) - \arctan(2x)}{x^2}\,dx$$

$$= \int \left(5\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}(-1)^{n+1}\right) - \left(\frac{2}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}\,2^{2n+1}}{2n+1}\right.$$

integrate

$$= 5\sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)}(-1)^{n+1}$$

$$- 2\ln(x) - \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}\,2^{2n+1}}{(2n+1)(2n)} + C$$

Example 2 (Hard)

$$\frac{2x^3}{(1+x^2)^5}$$

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

derivative

$$(1-x)^{-2} = \sum_{n=0}^{\infty} n x^{n-1}$$

derivative

$$2(1-x)^{-3} = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

Keep on going, plug in $-x^2$

Example 3: Find the sum:

$$\sum_{n=1}^{\infty} \frac{\pi^{2n+1}}{(2n)!\, 36^n}$$

Relate to a MacLaurin series

$$= \pi \sum_{n=1}^{\infty} \frac{\pi^{2n}}{(2n)!\,(6^2)^n}$$

$$= \pi \sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$$

$$\pi \sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

so close

but not enough!

no $(-1)^n$ in given series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$e^{x} + e^{-x}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$\overbrace{1 + \cancel{x} + \frac{x^2}{2} + \cancel{\frac{x^3}{6}} + \cdots} + \overbrace{1 - \cancel{x} + \frac{x^2}{2} - \cancel{\frac{x^3}{6}} + \cdots}$$

$$= 2\left(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)$$

$$= 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

So $\dfrac{e^x + e^{-x}}{2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, then

$= \cosh(x)$

$$\frac{e^x + e^{-x}}{2} - 1 = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Our series was $\pi \displaystyle\sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$,

$$= \boxed{\pi\left(\frac{e^{\pi/6} + e^{-\frac{\pi}{6}}}{2} - 1\right)}$$

Example 4.

Find the sum:

$$\sum_{n=1}^{\infty} \frac{(-1/2)^n}{n}$$

what is on the denominator?

ln(x) looks like this

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{n} (-1)^{n+1}$$

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{n}(-1)^{n+1}$$

$x-1=\frac{1}{2}, x=\frac{3}{2}$

$$\ln\left(\frac{3}{2}\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n}(-1)^{n+1}$$

$$= -\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n}$$

so

$$-\ln\left(\frac{3}{2}\right) = \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n}$$

List of Common Taylor Series

MacLaurin

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$$ (radius 1)

$$\underbrace{\ln(x)}_{\text{not MacLaurin}} = \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n}$$ (radius 1)

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$ (radius 1)

MacLaurin

All Maclaurin

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ (radius ∞)

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$ (radius ∞)

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$